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ON GAUSS-KRONROD QUADRATURE FORMULAE OF CHEBYSHEV TYPE

SOTIRIOS E. NOTARIS

ABSTRACT. We prove that there is no positive measure $d\sigma$ on (a, b) such that the corresponding Gauss-Kronrod quadrature formula is also a Chebyshev quadrature formula. The same is true if we consider measures of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, on a symmetric interval $(-a, a)$, and the Gauss-Kronrod formula is required to have equal weights only for n even. We also show that the only positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$ for which the Gauss-Kronrod formula has all weights equal if $n = 1$, or has the form $\int_{-1}^{1} f(t)\,d\sigma(t) = w\sum_{\nu=1}^{n} f(\tau_\nu) + w_1 f(1) + w\sum_{\mu=2}^{n} f(\tau_\mu^*) + w_1 f(-1) + R_n^K(f)$ for all $n \geq 2$, is the Chebyshev measure of the first kind $d\sigma_C(t) = (1-t^2)^{-1/2}dt$.

1. INTRODUCTION

Let $d\sigma$ be a positive measure on the interval (a, b), whose moments all exist,

$$(1.1) \qquad \mu_i = \int_a^b t^i\,d\sigma(t) < \infty, \qquad i = 0, 1, 2, \ldots.$$

The Gauss-Kronrod quadrature formula for $d\sigma$ has the form

$$(1.2) \qquad \int_a^b f(t)\,d\sigma(t) = \sum_{\nu=1}^{n} \sigma_\nu f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^* f(\tau_\mu^*) + R_n^K(f),$$

where $\tau_\nu = \tau_\nu^{(n)}$ are the zeros of the nth-degree (monic) orthogonal polynomial $\pi_n(\cdot) = \pi_n(\cdot\,; d\sigma)$, and the $\tau_\mu^* = \tau_\mu^{*(n)}$, $\sigma_\nu = \sigma_\nu^{(n)}$, and $\sigma_\mu^* = \sigma_\mu^{*(n)}$ are determined such that (1.2) has maximum degree of exactness (at least) $3n+1$, i.e., $R_n^K(f) = 0$ for all $f \in \mathbb{P}_{3n+1}$. Then the τ_μ^* must be the zeros of a (monic) polynomial $\pi_{n+1}^*(\cdot) = \pi_{n+1}^*(\cdot\,; d\sigma)$, called the Stieltjes polynomial, which satisfies the orthogonality condition

$$(1.3) \qquad \int_a^b \pi_n(t)\pi_{n+1}^*(t)t^i\,d\sigma(t) = 0, \qquad i = 0, 1, \ldots, n,$$

that is, π_{n+1}^* is orthogonal to all polynomials of lower degree with respect to the oscillatory measure $d\sigma^*(t) = \pi_n(t)d\sigma(t)$ on (a, b). It can be shown that

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π_{n+1}^* is uniquely defined by (1.3) (see [1, §4]). However, since $d\sigma^*$ changes sign on (a, b), the reality of the τ_μ^* cannot in general be assumed.

A quadrature rule

$$(1.4)\qquad \int_a^b f(t)\,d\sigma(t) = \sum_{i=1}^n w_i f(t_i) + R_n^C(f)$$

with equal weights

$$(1.5)\qquad w_1^{(n)} = w_2^{(n)} = \cdots = w_n^{(n)}$$

is called a Chebyshev quadrature rule, if the nodes $t_k = t_k^{(n)}$ are real and if (1.4) has degree of exactness (at least) n. By setting $f(t) = 1$ in (1.4), we find, in view of (1.5),

$$(1.6)\qquad w_i = \frac{\mu_0}{n}, \qquad i = 1, 2, \ldots, n.$$

It is well known that the only equally weighted (for all n) Gauss formula is the one relative to the Chebyshev measure of the first kind, $d\sigma_C(t) = (1-t^2)^{-1/2}dt$ on $(-1, 1)$ (see, e.g., [2, §4]). If the Gauss formula has the equicoefficient property only for n even, then among the positive measures of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, with symmetric support, the only one admitting such a formula is, up to a linear transformation,

$$(1.7)\qquad d\sigma_\xi(t) = \begin{cases} |t|(t^2-\xi^2)^{-1/2}(1-t^2)^{-1/2}dt, & t \in [-1, -\xi] \cup [\xi, 1], \\ 0 & \text{elsewhere} \end{cases}$$

(see [3, §6]).

The only Gauss-Kronrod formula known, which is almost of Chebyshev type, is the one relative to the Chebyshev measure of the first kind,

$$(1.8)\qquad \begin{aligned} \int_{-1}^1 f(t)(1-t^2)^{-1/2}\,dt &= \frac{\pi}{3}\sum_{i=1}^3 f\left(\cos\frac{2i-1}{6}\pi\right) + R_1^{KC}(f), \\ \int_{-1}^1 f(t)(1-t^2)^{-1/2}\,dt &= \frac{\pi}{2n}\left[\frac{1}{2}f(-1) + \sum_{i=1}^{2n-1} f\left(\cos\frac{i\pi}{2n}\right) + \frac{1}{2}f(1)\right] \\ &\quad + R_n^{KC}(f), \qquad n \ge 2 \end{aligned}$$

(see, e.g., [6, equation (43)]). Incidentally, the second formula in (1.8) is the same as the Gauss-Lobatto formula for $d\sigma_C(t) = (1-t^2)^{-1/2}dt$ with $2n+1$ points, hence it has elevated degree of exactness $4n-1$.

Equally-weighted quadrature rules are useful in practice because they minimize the effect of random errors in the values of $f(t_i)$ in (1.4). Therefore, it is interesting to examine if there are positive measures admitting Gauss-Kronrod formulae of Chebyshev type. In the next section we show that such measures do not exist. This is also the case if we consider measures of the form $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, with symmetric support, and the Gauss-Kronrod formula is required to have equal weights only for n even. Naturally, one then wonders if there are at least other Gauss-Kronrod formulae of the form (1.8). In §3, we prove that the only positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$, which gives rise to a Gauss-Kronrod formula of the type (1.8), is $d\sigma_C(t) = (1-t^2)^{-1/2}dt$. In both sections we follow the technique used by

Geronimus [4, 5] to show that the only Gauss formula of Chebyshev type is the one relative to the Chebyshev measure of the first kind.

2. Nonexistence of Gauss-Kronrod quadrature formulae of Chebyshev type

It is known that for any positive measure $d\sigma$ on (a, b) the respective (monic) orthogonal polynomials $\{\pi_i(\cdot\,; d\sigma)\}$ satisfy a three-term recurrence relation of the form

$$\begin{aligned} &\pi_{-1}(t) = 0, \qquad \pi_0(t) = 1, \\ &\pi_{i+1}(t) = (t-\alpha_i)\pi_i(t) - \beta_i\pi_{i-1}(t), \qquad i = 0, 1, 2, \dots, \end{aligned} \tag{2.1}$$

where the recursion coefficients $\alpha_i = \alpha_i(d\sigma)$ and $\beta_i = \beta_i(d\sigma)$ are given by the formulae

$$\begin{aligned} &\alpha_i = \frac{\int_a^b t[\pi_i(t)]^2\,d\sigma(t)}{\int_a^b [\pi_i(t)]^2\,d\sigma(t)}, \qquad i = 0, 1, 2, \dots, \\ &\beta_0 = \int_a^b d\sigma(t), \qquad \beta_i = \frac{\int_a^b [\pi_i(t)]^2\,d\sigma(t)}{\int_a^b [\pi_{i-1}(t)]^2\,d\sigma(t)}, \qquad i = 1, 2, \dots, \end{aligned} \tag{2.2}$$

hence $\beta_i > 0$, $i = 0, 1, 2, \dots$. Using (2.1) and induction, we can show that

$$\begin{aligned} \pi_n(t) = t^n &- \left(\sum_{i=0}^{n-1}\alpha_i\right)t^{n-1} \\ &+ \left(\sum_{\substack{i,j=0\\ i<j}}^{n-1}\alpha_i\alpha_j - \sum_{i=1}^{n-1}\beta_i\right)t^{n-2} + \cdots, \qquad n \geq 1. \end{aligned} \tag{2.3}$$

A similar formula can be obtained for the corresponding Stieltjes polynomial.

Lemma 2.1. *The Stieltjes polynomial* $\pi_{n+1}^*(\cdot\,; d\sigma)$ *has the form*

$$\begin{aligned} \pi_{n+1}^*(t) = t^{n+1} &- \left(\sum_{i=0}^{n}\alpha_i\right)t^{n} \\ &+ \left(\sum_{\substack{i,j=0\\ i<j}}^{n}\alpha_i\alpha_j - \sum_{i=1}^{n+1}\beta_i\right)t^{n-1} + \cdots, \qquad n \geq 1. \end{aligned} \tag{2.4}$$

Proof. Expanding π_{n+1}^* in terms of π_i, we have

$$\pi_{n+1}^*(t) = \pi_{n+1}(t) + c_0\pi_n(t) + \cdots + c_{n-1}\pi_1(t) + c_n\pi_0(t), \tag{2.5}$$

and then substituting into (1.3) with $i = 0, 1$ yields, by means of (2.2) and orthogonality,

$$c_0 = 0, \qquad c_1 = -\beta_{n+1}. \tag{2.6}$$

These, together with (2.3) and (2.5), imply (2.4). □

We can now prove our main result.

Theorem 2.2. *There is no positive measure $d\sigma$ on (a, b) relative to which (1.2) is also a Chebyshev quadrature formula for each $n = 1, 2, \ldots$.*

Proof. Assume that there exists a positive measure $d\sigma$ on (a, b) for which (1.2) is a Chebyshev quadrature formula, that is, has the form

$$\int_a^b f(t)\,d\sigma(t) = \frac{\mu_0}{2n+1}\left[\sum_{\nu=1}^{n} f(\tau_\nu) + \sum_{\mu=1}^{n+1} f(\tau_\mu^*)\right] + R_n^K(f). \tag{2.7}$$

Since for each $n = 1, 2, \ldots$, (2.7) is exact for $f(t) = t$ and $f(t) = t^2$, we obtain

$$\sum_{\nu=1}^{n} \tau_\nu + \sum_{\mu=1}^{n+1} \tau_\mu^* = (2n+1)m_1, \qquad \sum_{\nu=1}^{n} \tau_\nu^2 + \sum_{\mu=1}^{n+1} \tau_\mu^{*2} = (2n+1)m_2^2, \tag{2.8}$$

where $m_1 = \mu_1/\mu_0$ and $m_2^2 = \mu_2/\mu_0$. Let

$$p_{2n+1}(t) = \pi_n(t)\pi_{n+1}^*(t) \tag{2.9}$$

with $\pi_n(t) = \prod_{\nu=1}^{n}(t - \tau_\nu)$ and $\pi_{n+1}^*(t) = \prod_{\mu=1}^{n+1}(t - \tau_\mu^*)$. First, because of (2.8) we must have

$$p_{2n+1}(t) = t^{2n+1} - (2n+1)m_1 t^{2n} + \frac{2n+1}{2}\left[(2n+1)m_1^2 - m_2^2\right] t^{2n-1} + \cdots. \tag{2.10}$$

Also, substituting $\pi_n(t)$ and $\pi_{n+1}^*(t)$ in (2.9) from (2.3) and (2.4), one finds, after a simple computation,

$$\begin{aligned} p_{2n+1}(t) = {}& t^{2n+1} - \left(2\sum_{i=0}^{n-1}\alpha_i + \alpha_n\right)t^{2n} \\ & + \left[\left(\sum_{i=0}^{n-1}\alpha_i\right)^2 + 2\sum_{\substack{i,j=0\\ i<j}}^{n}\alpha_i\alpha_j - \left(2\sum_{i=1}^{n-1}\beta_i + \beta_n + \beta_{n+1}\right)\right]t^{2n-1} + \cdots. \end{aligned} \tag{2.11}$$

Equating the coefficients of t^{2n} and t^{2n-1} gives

$$\begin{aligned} & 2\sum_{i=0}^{n-1}\alpha_i + \alpha_n = (2n+1)m_1, \qquad n \ge 1, \\ & \left(\sum_{i=0}^{n-1}\alpha_i\right)^2 + 2\sum_{\substack{i,j=0\\ i<j}}^{n}\alpha_i\alpha_j - \left(2\sum_{i=1}^{n-1}\beta_i + \beta_n + \beta_{n+1}\right) \\ & \qquad = \frac{2n+1}{2}\left[(2n+1)m_1^2 - m_2^2\right], \qquad n \ge 1. \end{aligned} \tag{2.12}$$

Now from (2.2) we find

$$\alpha_0 = m_1, \qquad \beta_1 = m_2^2 - m_1^2, \tag{2.13}$$

which, inserted into (2.12), yields, for $n = 1$,

$$\alpha_1 = m_1, \qquad \beta_2 = \tfrac{1}{2}(m_2^2 - m_1^2), \tag{2.14}$$

and for $n = 2$,

$$\alpha_2 = m_1, \qquad \beta_3 = 0. \tag{2.15}$$

This contradicts the fact that $\beta_3 > 0$. □

The negative result of Theorem 2.2 leads us to explore the possibility of having Gauss-Kronrod formulae with equal weights only for n even. We restrict our search among the positive measures of the type $d\sigma(t) = \omega(t)dt$, where $\omega(t)$ is even, with symmetric support $(-a, a)$. Thus, we want (1.2) to have the form

$$\int_{-a}^{a} f(t)\omega(t)\,dt = \frac{\mu_0}{2n+1}\left[\sum_{\nu=1}^{n} f(\tau_\nu) + \sum_{\mu=1}^{n+1} f(\tau_\mu^*)\right] + R_n^K(f), \qquad n = 2k. \tag{2.16}$$

Since $d\sigma$ is an even measure with symmetric support, using orthogonality and (1.3), one easily shows by uniqueness that π_n and π_{n+1}^* are always either even or odd depending on the parity of n, that is,

$$\begin{aligned} \pi_n(-t) &= (-1)^n \pi_n(t), \qquad n = 0, 1, 2, \dots, \\ \pi_{n+1}^*(-t) &= (-1)^{n+1}\pi_{n+1}^*(t), \qquad n = 0, 1, 2, \dots. \end{aligned} \tag{2.17}$$

Consequently, the τ_ν and τ_μ^* in (2.16) are symmetric with respect to the origin. Setting $f(t) = g(t^2)$, $g \in \mathbb{P}_{[(3n+1)/2]}$, where $[\cdot]$ denotes the integer part of a real number, it follows by symmetry that

$$\int_0^a g(t^2)\omega(t)\,dt = \frac{\mu_0}{4k+1}\left[\sum_{\nu=1}^{k} g(\tau_\nu^2) + \sum_{\mu=1}^{k} g(\tau_\mu^{*2}) + \frac{1}{2}g(0)\right] \quad \text{for all } g \in \mathbb{P}_{3k}. \tag{2.18}$$

Letting $t = x^{1/2}$, so that $dt = \frac{1}{2}x^{-1/2}dx$, we get

$$\begin{aligned} &\int_0^{a^2} g(x)\omega(x^{1/2})x^{-1/2}\,dx \\ &= \frac{2\mu_0}{4k+1}\left[\sum_{\nu=1}^{k} g(\tau_\nu^2) + \sum_{\mu=1}^{k} g(\tau_\mu^{*2}) + \frac{1}{2}g(0)\right] \quad \text{for all } g \in \mathbb{P}_{3k}. \end{aligned} \tag{2.19}$$

If $\overline{\omega}(x) = \omega(x^{1/2})x^{-1/2}$ and $\overline{a} = a^2$, then $\overline{\mu}_0 = \mu_0$, $\overline{\tau}_\nu = \tau_\nu^2$, and $\overline{\tau}_\mu^* = \tau_\mu^{*2}$, where for the rest of this section all the quantities carrying a bar refer to the measure $d\overline{\sigma}(x) = \overline{\omega}(x)dx$ on $(0, \overline{a})$. Replacing x by t in (2.19), we obtain

$$\int_0^{\overline{a}} g(t)\overline{\omega}(t)\,dt = \frac{\overline{\mu}_0}{2k+1/2}\left[\sum_{\nu=1}^{k} g(\overline{\tau}_\nu) + \sum_{\mu=1}^{k} g(\overline{\tau}_\mu^*) + \frac{1}{2}g(0)\right] \quad \text{for all } g \in \mathbb{P}_{3k}. \tag{2.20}$$

Therefore, it comes down to examining if there exist Gauss-Kronrod formulae of the type (2.20), that is, one of the zeros of $\overline{\pi}_{k+1}^*$ is 0, and all the weights are equal except the one corresponding to the node at 0.

Our findings are given in the following:

Theorem 2.3. *There is no positive measure of the form* $d\sigma(t) = \omega(t)dt$, *where* $\omega(t)$ *is even, with symmetric support* $(-a, a)$, *for which* (1.2) *has equal weights for all* n *even.*

Proof. To prove the theorem, it suffices to show that there is no Gauss-Kronrod formula of the type (2.20). If we assume that such a formula exists, by much the same way as in the proof of Theorem 2.2, we find that $\overline{p}_{2k+1}$ (cf. (2.9)) must have the form

$$\begin{aligned}\overline{p}_{2k+1}(t) = t^{2k+1} - \left(2k+\frac{1}{2}\right)\overline{m}_1 t^{2k} \\ + \frac{2k+1/2}{2}\left[\left(2k+\frac{1}{2}\right)\overline{m}_1^2 - \overline{m}_2^2\right] t^{2k-1} + \cdots ,\end{aligned} \tag{2.21}$$

where $\overline{m}_1 = \overline{\mu}_1/\overline{\mu}_0$ and $\overline{m}_2^2 = \overline{\mu}_2/\overline{\mu}_0$. Then from (2.21) and (2.11), with $\overline{p}_{2k+1}$ in place of p_{2n+1}, we obtain the equations

$$\begin{aligned} 2\sum_{i=0}^{k-1}\overline{\alpha}_i + \overline{\alpha}_k = \left(2k+\frac{1}{2}\right)\overline{m}_1, \qquad k \ge 1, \\ \left(\sum_{i=0}^{k-1}\overline{\alpha}_i\right)^2 + 2\sum_{\substack{i,j=0\\ i<j}}^{k}\overline{\alpha}_i\overline{\alpha}_j - \left(2\sum_{i=1}^{k-1}\overline{\beta}_i + \overline{\beta}_k + \overline{\beta}_{k+1}\right) \\ = \frac{2k+1/2}{2}\left[\left(2k+\frac{1}{2}\right)\overline{m}_1^2 - \overline{m}_2^2\right], \qquad k \ge 1,\end{aligned} \tag{2.22}$$

where $\overline{\alpha}_i = \alpha_i(d\overline{\sigma})$ and $\overline{\beta}_i = \beta_i(d\overline{\sigma})$. From (2.13) we have

$$\overline{\alpha}_0 = \overline{m}_1, \qquad \overline{\beta}_1 = \overline{m}_2^2 - \overline{m}_1^2. \tag{2.23}$$

Then (2.22) gives, for $k = 1$,

$$\overline{\alpha}_1 = \tfrac{1}{2}\overline{m}_1, \qquad \overline{\beta}_2 = \tfrac{1}{4}(\overline{m}_2^2 - \tfrac{1}{2}\overline{m}_1^2), \tag{2.24}$$

and for $k = 2$,

$$\overline{\alpha}_2 = \tfrac{3}{2}\overline{m}_1, \qquad \overline{\beta}_3 = -\tfrac{1}{4}\overline{m}_1^2, \tag{2.25}$$

which contradicts the fact that $\overline{\beta}_3 > 0$. □

3. Gauss-Kronrod quadrature formulae almost of Chebyshev type

Throughout this section $d\sigma$ is a positive and even measure $d\sigma(t) = d\sigma(-t)$ on $(-1, 1)$. Then some of the quantities and formulae of the previous sections take a special form. First, it is clear from (1.1) that

$$\mu_i = 0 \quad \text{for all } i \text{ odd.} \tag{3.1}$$

Also, π_i is either even or odd depending on the parity of i (cf. (2.17)). Then it follows from (2.2) that

$$\alpha_i = 0, \qquad i = 0, 1, 2, \ldots . \tag{3.2}$$

Moreover, (2.4) with $n = 1$ implies

$$\pi_2^*(t) = t^2 - (\beta_1 + \beta_2), \tag{3.3}$$

hence (cf. (2.9) with $n = 1$)

$$p_3(t) = \pi_1(t)\pi_2^*(t) = t^3 - (\beta_1 + \beta_2)t = \pi_3(t), \tag{3.4}$$

where for the last equality in (3.4) we used (2.3) and (2.17). Therefore, for $n = 1$, the Gauss-Kronrod formula is the 3-point Gauss formula. (The same is true if the support of $d\sigma$ is any interval symmetric with respect to the origin.) We want to determine if there are any other positive and even measures $d\sigma$ on $(-1, 1)$, besides the Chebyshev measure of the first kind, for which the Gauss-Kronrod formula has the form

$$\begin{aligned} &\int_{-1}^{1} f(t)\,d\sigma(t) = \frac{\mu_0}{3}[f(\tau_1) + f(\tau_1^*) + f(\tau_2^*)] + R_1^K(f), \\ &\int_{-1}^{1} f(t)\,d\sigma(t) = w\sum_{\nu=1}^{n} f(\tau_\nu) + w_1 f(-1) \\ &\qquad + w\sum_{\mu=2}^{n} f(\tau_\mu^*) + w_1 f(1) + R_n^K(f), \qquad n \ge 2, \end{aligned} \tag{3.5}$$

that is, for all $n \ge 2$ two of the zeros of π_{n+1}^* are ± 1, and all the weights are equal except those corresponding to the nodes at ± 1.

The existence of quadrature formulae of this kind is described in the following:

Theorem 3.1. *The only positive and even measure $d\sigma$ on $(-1, 1)$ for which the Gauss-Kronrod quadrature formula has the form (3.5) is the Chebyshev measure of the first kind $d\sigma_C(t) = (1 - t^2)^{-1/2}dt$.*

Proof. We proceed along the lines of the proof of Theorem 2.2. Assume that for the positive and even measure $d\sigma$ on $(-1, 1)$ the Gauss-Kronrod formula is of the type (3.5). First, (3.1) implies that $m_1 = \mu_1/\mu_0 = 0$. Then from (2.13) we find

$$\beta_1 = m_2^2, \tag{3.6}$$

where $m_2^2 = \mu_2/\mu_0$, and for $n = 1$, as in the proof of Theorem 2.2 (cf. (2.14)), we get

$$\beta_2 = \tfrac{1}{2}m_2^2. \tag{3.7}$$

If $n \ge 2$, since $d\sigma$ is an even measure with symmetric support, the τ_ν and τ_μ^* in the second formula in (3.5) are symmetric with respect to the origin. Also, this formula is exact for $f(t) = 1$ and $f(t) = t^2$; hence we obtain, after setting $w_1 = cw$,

$$\begin{aligned} &(2n + 2c - 1)w = \mu_0, \\ &w\left(\sum_{\nu=1}^{n} \tau_\nu^2 + \sum_{\mu=2}^{n} \tau_\mu^{*2} + 2c\right) = \mu_2, \end{aligned} \tag{3.8}$$

from which it follows that

$$\sum_{\nu=1}^{n} \tau_\nu^2 + \sum_{\mu=2}^{n} \tau_\mu^{*2} + 2 = (2n+2c-1)m_2^2 + 2(1-c). \tag{3.9}$$

Because of this, p_{2n+1} (cf. (2.9)) must have the form

$$p_{2n+1}(t) = t^{2n+1} - \left[\left(n+c-\tfrac{1}{2}\right) m_2^2 + 1 - c\right] t^{2n-1} + \cdots . \tag{3.10}$$

Moreover, from (2.11) and (3.2) we have

$$p_{2n+1}(t) = t^{2n+1} - \left(2\sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1}\right) t^{2n-1} + \cdots . \tag{3.11}$$

By equating the coefficients of t^{2n-1}, we derive the equation

$$2\sum_{i=1}^{n-1} \beta_i + \beta_n + \beta_{n+1} = \left(n+c-\tfrac{1}{2}\right) m_2^2 + 1 - c, \qquad n \geq 2. \tag{3.12}$$

Applying (3.12) for two successive values of n, and then subtracting the two equations, we get

$$\beta_n + \beta_{n+2} = m_2^2, \qquad n \geq 2, \tag{3.13}$$

which, by means of (3.7), gives

$$\beta_{2j} = \tfrac{1}{2} m_2^2, \qquad j = 1, 2, \ldots . \tag{3.14}$$

Now using (1.3) with $i = 2, 3$, (2.1), and orthogonality, we find, after a lengthy but straightforward computation, that c_2 and c_3 in (2.5), when $d\sigma$ is an even measure, are given by

$$c_2 = 0, \qquad c_3 = (\beta_{n-1} - \beta_{n+2})\beta_{n+1}. \tag{3.15}$$

Then from (2.5), (2.6), (3.15), and (2.1) we get analytic expressions for π_3^* and π_4^*,

$$\begin{aligned} \pi_3^*(t) &= t^3 - (\beta_1 + \beta_2 + \beta_3)t, \\ \pi_4^*(t) &= t^4 - (\beta_1 + \beta_2 + \beta_3 + \beta_4)t^2 + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5. \end{aligned} \tag{3.16}$$

Since $\pi_3^*(\pm 1) = 0$ and $\pi_4^*(\pm 1) = 0$, we obtain the equations

$$\begin{aligned} &1 - (\beta_1 + \beta_2 + \beta_3) = 0, \\ &1 - (\beta_1 + \beta_2 + \beta_3 + \beta_4) + \beta_1\beta_3 + \beta_1\beta_4 + \beta_2\beta_4 - \beta_4\beta_5 = 0. \end{aligned} \tag{3.17}$$

These, together with

$$\beta_1 = 2\beta_2 = 2\beta_4 \tag{3.18}$$

(cf. (3.6) and (3.14)), yield

$$\beta_4(\beta_3 - \beta_5) = 0, \tag{3.19}$$

which, on account of $\beta_4 > 0$, gives

$$\beta_3 = \beta_5. \tag{3.20}$$

Then (3.13) implies

$$\beta_{2j+1} = \tfrac{1}{2} m_2^2, \qquad j = 1, 2, \ldots , \tag{3.21}$$

so finally

(3.22) $$\beta_1 = m_2^2, \qquad \beta_i = \tfrac{1}{2}m_2^2, \quad i = 2, 3, \ldots .$$

Substituting β_1, β_2, β_3 from (3.22) into the first equation in (3.17), we obtain

(3.23) $$m_2^2 = \tfrac{1}{2}.$$

Therefore,

(3.24) $$\beta_1 = \tfrac{1}{2}, \qquad \beta_i = \tfrac{1}{4}, \quad i = 2, 3, \ldots ,$$

and $d\sigma$ is the Chebyshev measure of the first kind. □

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Department of Mathematical Sciences, Indiana University–Purdue University at Indianapolis, 1125 E. 38th Street, Indianapolis, Indiana 46205

Current address: Department of Mathematics, University of Missouri–Columbia, Columbia, Missouri 65211

E-mail address: sotirios@notaris.cs.missouri.edu